

Quasilinear elliptic systems with convex-concave singular terms Φ -Laplacian operator *

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Abstract

This paper deals with existence of positive solutions for a class of quasilinear elliptic systems involving the Φ -Laplacian operator and convex-concave singular terms. Our approach is based on the generalized Galerkin Method along with perturbation techniques and comparison arguments in the setting of Orlicz-Sobolev spaces.

Key Words: Elliptic singular systems, Comparison principle, Galerkin-type method, Φ -Laplacian operator, Orlicz-Sobolev spaces.

1 Introduction

This paper deals with the existence of solutions of elliptic systems of the form

$$\begin{cases} -\Delta_{\Phi} u = \frac{a_1(x)}{u^{\alpha_1} v^{\beta_1}} + b_1(x) u^{\gamma_1} v^{\sigma_1} & \text{in } \Omega, \\ -\Delta_{\Phi} v = \frac{a_2(x)}{u^{\beta_2} v^{\alpha_2}} + b_2(x) u^{\sigma_2} v^{\gamma_2} & \text{in } \Omega, \\ u, v > 0 & \text{in } \Omega, \quad u = v = 0 \text{ on } \partial\Omega, \end{cases} \quad (1.1)$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$; $\alpha_i, \beta_i, \gamma_i, \sigma_i \geq 0$ are real constants; and $a_i, b_i : \Omega \rightarrow \mathbb{R}$ are non-negative, measurable functions. In addition, Φ is the N-function defined by

$$\Phi(t) = \int_0^t s \phi(|s|) ds, \quad t \in \mathbb{R},$$

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where $\phi : (0, \infty) \rightarrow (0, \infty)$ is C^1 and satisfies:

$$(\phi_1) \quad (i) \quad s\phi(s) \rightarrow 0 \text{ as } s \rightarrow 0, \quad (ii) \quad s\phi(s) \rightarrow \infty \text{ as } s \rightarrow \infty$$

$$(\phi_2) \quad s \mapsto s\phi(s) \text{ is strictly increasing in } (0, \infty),$$

$$(\phi_3) \quad \text{there exist } \ell, m \in (1, N) \text{ such that}$$

$$\ell - 1 \leq \frac{(s\phi(s))'}{\phi(s)} \leq m - 1, \quad s > 0,$$

and $-\Delta_\Phi : W_0^{1,\Phi}(\Omega) \rightarrow W^{-1,\tilde{\Phi}}(\Omega)$ is defined by

$$\langle -\Delta_\Phi u, v \rangle := \int_\Omega \phi(|\nabla u|) \nabla u \nabla v dx, \quad u, v \in W_0^{1,\Phi}(\Omega),$$

where $W_0^{1,\Phi}(\Omega)$ stands for the classical Orlicz-Sobolev space, $W^{-1,\tilde{\Phi}}(\Omega)$ denotes its dual space, and $\tilde{\Phi}$, given by

$$\tilde{\Phi}(t) = \max_{s \geq 0} \{ts - \Phi(s)\}, \quad t \geq 0,$$

is the N-function complementary to the N-function Φ and vice-versa. We refer the reader to Section 5 for more details about this space and about the Orlicz spaces that will be denoted by $L_\Phi(\Omega)$.

In this context, we prove an existence result of positive solutions to Problem (1.1) and a weak Comparison Principle for the Φ -Laplacian operator. In this work, $d(x) = \inf\{|x-y| \mid y \in \partial\Omega\}$ for $x \in \Omega$ will stand for the distance function to the boundary of the domain Ω .

Theorem 1.1 *Assume $(\phi_1) - (\phi_3)$, $0 \neq a_i \in L_{\tilde{\Psi}}(\Omega)$, $0 \neq b_i \in L^{q_i}(\Omega)$, and $a_i + b_i > 0$ a.e. in Ω hold for some N-function $\Phi < \Psi < \Phi_*$ and $q_i \geq \ell/(\ell - \sigma_i - \gamma_i - 1)$, where $0 < \sigma_i + \gamma_i < \ell - 1$. Suppose in addition that $a_i d^{-\alpha_i - \beta_i} \in L_{\tilde{\Psi}}(\Omega)$. Then there exists $(u, v) \in W_0^{1,\Phi}(\Omega) \times W_0^{1,\Phi}(\Omega)$ weak solution of (1.1) if one of the below condition is true:*

$$(i) \quad \beta_i = 0 \text{ (cooperative structure),}$$

$$(ii) \quad \sigma_i = 0 \text{ (non-cooperative structure),}$$

$$(iii) \quad \alpha_i = \gamma_i = 0, \text{ and } \min\{a_i, b_i\} > 0 \text{ a.e. } x \in \Omega \text{ (mixed structure).}$$

Besides this, there exists a $C > 0$ such that $u(x), v(x) \geq Cd(x)$ a.e. $x \in \Omega$.

Remark 1.1 *The item (i) is true if we assume $0 < \alpha_i \leq 1$ instead of $a_i d^{-\alpha_i} \in L_{\tilde{\Psi}}(\Omega)$.*

One important tool in our approach is the comparison principle below, which is relevant by itself. Consider the problem

$$\begin{cases} -\Delta_\Phi u = f(x, u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, u = 0 \text{ on } \partial\Omega, \end{cases} \quad (1.2)$$

where $f : \Omega \times [0, \infty) \rightarrow \mathbb{R}$ is a Carathéodory function.

Definition 1.1 An $u \in W^{1,\Phi}(\Omega)$ is a subsolution [solution] (supersolution) of (1.2) if

$$\int_{\Omega} \phi(|\nabla u|) \nabla u \nabla \varphi dx \leq [=](\geq) \int_{\Omega} f(x, u) \varphi dx, \quad \varphi \in W_0^{1,\Phi}(\Omega), \quad \varphi \geq 0.$$

Theorem 1.2 (Comparison Principle) Let ϕ be satisfy $(\phi_1) - (\phi_3)$. Assume that

$$t \mapsto \frac{f(x, t)}{t^{\ell-1}} \text{ is decreasing for a.e. } x \in \Omega.$$

If $u_1, u_2 \in W^{1,\Phi}(\Omega)$ are sub and supersolution of (1.2), respectively, such that $u_1/u_2 \in L^\infty(\Omega)$ and $u_1 \leq u_2$ in $\partial\Omega$, then $u_1 \leq u_2$ a.e. in Ω .

This result extend and improve to $\Phi - Laplacian$ operator results that are well-known to $Laplacian$ (Brézis and Oswald [3]) and $p - Laplacian$ (Diaz and Saa [9] and Mohammed [25]) operators.

Below, let us do an overview about related problems to (1.1). First, we point out that there is by now an extensive literature on single-equation singular problems related to (1.1) ($\phi(t) = t^{p-2}$ with $1 < p < N$), that is, to problems like

$$\begin{cases} -\Delta_p u = \frac{a(x)}{u^\alpha} + b(x)u^\gamma & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \end{cases}$$

where a, b are appropriate potentials and α, γ are positive real constants. As it is impossible to cite all important papers that have considered this kind of problems, let us to refer the reader to works [8, 14, 16, 17, 23, 25], and their references to highlight the variety of techniques that have been used to solve them.

Again, for the function $\phi(t) = t^{p-2}$ with $1 < p < N$, we note that Problem (1.1) read as

$$\begin{cases} -\Delta_p u = \frac{a_1(x)}{u^{\alpha_1} v^{\beta_1}} + b_1(x) u^{\gamma_1} v^{\sigma_1} & \text{in } \Omega, \\ -\Delta_p v = \frac{a_2(x)}{u^{\beta_2} v^{\alpha_2}} + b_2(x) u^{\sigma_2} v^{\gamma_2} & \text{in } \Omega, \\ u, v > 0 & \text{in } \Omega, \quad u = v = 0 \text{ on } \partial\Omega, \end{cases} \quad (1.3)$$

and the assumptions $(\phi_1) - (\phi_3)$ are true with $\ell = m = p$. When $p = 2$, problems like these have been considered with more frequency. We quote [2, 6, 13, 20, 21, 27], and references therein. About more general operators related to Problem (1.1), we refer the reader to [22] and their references. An important and recent paper in this context is [15]. In it the authors considered a $(p, q) - Laplacian$ system and proved existence and uniqueness results for $b_i = 0$ and $a_i = 1$ in (1.3) by using monotonicity methods.

Other classes of functions ϕ , which satisfy $(\phi_1) - (\phi_3)$ are:

- (i) $\phi(t) = t^{p-2} + t^{q-2}$ with $1 < p < q < N$. In this case, the problem (1.1) becomes in the (p, q) -Laplacian problem

$$\begin{cases} -\Delta_p u - \Delta_q u = \frac{a_1(x)}{u^{\alpha_1} v^{\beta_1}} + b_1(x) u^{\gamma_1} v^{\sigma_1} & \text{in } \Omega, \\ -\Delta_p v - \Delta_q v = \frac{a_2(x)}{u^{\beta_2} v^{\alpha_2}} + b_2(x) u^{\sigma_2} v^{\gamma_2} & \text{in } \Omega, \\ u, v > 0 & \text{in } \Omega, \quad u = v = 0 \text{ on } \partial\Omega, \end{cases}$$

with $\ell = p$ and $m = q$ in the assumption (ϕ_3) ,

(ii) $\phi(t) = \sum_{i=1}^N t^{p_i-2}$, where $1 < p_1 < p_2 < \dots < p_N$, and $\frac{1}{\bar{p}} = \frac{1}{N} \sum_{i=1}^N \frac{1}{p_i}$ with $\bar{p} < N$, $\ell = p_1$, and $m = p_N$ in the assumption (ϕ_3) . In this case, the corresponding problem is

$$\begin{cases} -\sum_{i=1}^N \Delta_{p_i} u = \frac{a_1(x)}{u^{\alpha_1} v^{\beta_1}} + b_1(x) u^{\gamma_1} v^{\sigma_1} & \text{in } \Omega, \\ -\sum_{i=1}^N \Delta_{p_i} v = \frac{a_2(x)}{u^{\beta_2} v^{\alpha_2}} + b_2(x) u^{\sigma_2} v^{\gamma_2} & \text{in } \Omega, \\ u, v > 0 & \text{in } \Omega, \quad u = v = 0 \text{ on } \partial\Omega, \end{cases}$$

which is known as an anisotropic elliptic problem in the literature.

The proof of our Theorems are organized in three sections. In section 2, we prove Theorem 1.2. The principal difficulty that we found it has been to prove the convexity of an operator, which was defined inspired in one introduced by Diaz & Saa [9]. Thanks to assumption (ϕ_3) , we were able to show this. Also, with an accurate analysis, we removed the hypothesis $u_2/u_1 \in L^\infty(\Omega)$ that was considered in former results to p -Laplacian operator. This was crucial in our approach.

In section 3, we consider one “regularization” of problem (1.1), and we proved Theorem 3.1 that has mathematical interest by itself. In particular, we proved in this theorem that the problem

$$\begin{cases} -\Delta_\Phi u = \frac{a_1(x)}{(u+1)^{\alpha_1} (v+1)^{\beta_1}} + b_1(x) u^{\gamma_1} v^{\sigma_1}, & \text{in } \Omega, \\ -\Delta_\Phi v = \frac{a_2(x)}{(u+1)^{\beta_2} (v+1)^{\alpha_2}} + b_2(x) u^{\sigma_2} v^{\gamma_2} & \text{in } \Omega, \\ u, v \geq 0 & \text{in } \Omega, \quad u = v = 0 \text{ on } \partial\Omega \end{cases}$$

admits a non-trivial solution in $W_0^{1,\Phi}(\Omega) \times W_0^{1,\Phi}(\Omega)$ for $0 \leq a_i, b_i \in L^\infty(\Omega)$ with $a_i + b_i \neq 0$, and $\alpha_i, \beta_i, \gamma_i, \sigma_i \geq 0$ with $\gamma_i + \sigma_i < \ell - 1$, $i = 1, 2$.

To prove Theorem 3.1, we developed an generalized Galerkin Method inspired in an idea found in the work of Browder [4] of 1983. We believe that this approach can be very useful principally to solve problems that do not have Variational structure. In particular, we can have Variational structure in Problem (1.1) if we require a number of relationship among the powers.

Finally, in section 4, we complete the proof of Theorem 1.1. Due our approach, we were able to consider cooperative (the right side of the first equation is increasing in $v > 0$ and the right side of the second equation is increasing in $u > 0$), non-cooperative and mixed structures with a few modifications in their proofs. The most important issue in this section is to show that the sequence obtained from Theorem 3.1 is bounded in $W_0^{1,\Phi}(\Omega) \times W_0^{1,\Phi}(\Omega)$ and it is also bounded from below by a positive function, namely, the distance function.

We also point out that as one motivation for us to prove the Theorem 1.1, it was the absence in literature of existence results of positive solutions to singular systems problems in Orlicz-Sobolev settings. Even in Sobolev settings for operators of the kind p -Laplacian with $p \neq 2$, existence results like Theorem 1.1 are not frequent.

2 Proof of Theorem 1.2

The proof is motivated by arguments in Díaz and Saa [9] .

Proof: To begin let $J : L^1(\Omega) \rightarrow (-\infty, \infty]$ be given by

$$J(u) := \begin{cases} \int_{\Omega} \Phi(|\nabla u^{1/\ell}|), & u \geq 0, u^{1/\ell} \in W_0^{1,\Phi}(\Omega), \\ \infty, & \text{otherwise.} \end{cases}$$

We claim that the efective domain of J is not empty, that is,

$$D(J) := \{u \in L^1(\Omega) \mid u \geq 0, u^{1/\ell} \in W_0^{1,\Phi}(\Omega)\} \neq \emptyset,$$

and, in particular, $J \neq \infty$. Indeed, given $x_0 \in \Omega$ take $\epsilon > 0$ such that $B_\epsilon(x_0) \subset \Omega$ ($B_\epsilon(x_0)$ is the ball centered at x_0 and radius $\epsilon > 0$) and consider the function

$$v_\epsilon(x) := \begin{cases} \bar{v}_\epsilon^s(|x - x_0|), & x \in B_\epsilon(x_0), \\ 0, & x \in \Omega - B_\epsilon(x_0), \end{cases}$$

where $\bar{v}_\epsilon : [0, \epsilon] \rightarrow \mathbb{R}$ is defined by

$$\bar{v}_\epsilon(t) := \begin{cases} 1, & t = 0, \\ \text{linear}, & 0 < t < \frac{\epsilon}{2}, \\ 0 & \frac{\epsilon}{2} \leq t \leq \epsilon \end{cases}$$

such that $s > \ell$.

So, by using Lemma 5.1 in the Appendix, we obtain

$$\begin{aligned} \int_{\Omega} \Phi(|\nabla(v_\epsilon)^{1/\ell}|) dx &= \int_A \Phi\left(\frac{s}{\ell} \bar{v}_\epsilon^{\frac{s}{\ell}-1} |\nabla \bar{v}_\epsilon|\right) dx \\ &\leq C \int_A \max\left\{\bar{v}_\epsilon^{(\frac{s}{\ell}-1)\ell}, \bar{v}_\epsilon^{(\frac{s}{\ell}-1)m}\right\} \Phi(|\nabla \bar{v}_\epsilon|) dx \\ &= C \int_A \bar{v}_\epsilon^{(\frac{s}{\ell}-1)\ell} \Phi(|\nabla \bar{v}_\epsilon|) dx \leq C \int_A \Phi(|\nabla \bar{v}_\epsilon|) dx < \infty. \end{aligned}$$

This shows our claim.

Below, let us show that J is a convex functional. Set $z_i := w_i^{1/\ell}$, $i = 1, 2$ and $z_3 = (\tau w_1 + (1 - \tau)w_2)^{1/\ell}$ for $w_1, w_2 \in D(J)$ and $\tau \in [0, 1]$. So, by applying the Hölder inequality, we get

$$\begin{aligned} z_3^{\ell-1} |\nabla z_3| &\leq \tau^{\frac{\ell-1}{\ell}} z_1^{\ell-1} \tau^{\frac{1}{\ell}} |\nabla z_1| + (1 - \tau)^{\frac{\ell-1}{\ell}} z_2^{\ell-1} (1 - \tau)^{\frac{1}{\ell}} |\nabla z_2| \\ &\leq (\tau z_1^\ell + (1 - \tau) z_2^\ell)^{\frac{\ell-1}{\ell}} (\tau |\nabla z_1|^\ell + (1 - \tau) |\nabla z_2|^\ell)^{\frac{1}{\ell}} \\ &= z_3^{\ell-1} (\tau |\nabla z_1|^\ell + (1 - \tau) |\nabla z_2|^\ell)^{\frac{1}{\ell}}. \end{aligned} \tag{2.1}$$

Besides this, by computing, we get

$$\begin{aligned}
\frac{d^2}{dt^2}\Phi(t^{1/\ell}) &= \frac{1}{\ell} \left[(s\phi(s))' \Big|_{s=t^{1/\ell}} \frac{1}{\ell} (t^{1/\ell-1})^2 + \phi(t^{1/\ell}) \left(\frac{1}{\ell} - 1 \right) (t^{1/\ell-1})^2 \right] \\
&= \frac{1}{\ell} (t^{1/\ell-1})^2 \phi(t^{1/\ell}) \left[\frac{(s\phi(s))'}{\phi(s)} \Big|_{s=t^{1/\ell}} \frac{1}{\ell} + \frac{1}{\ell} - 1 \right] \\
&\stackrel{(\phi_3)}{\geq} \frac{1}{\ell} (t^{1/\ell-1})^2 \phi(t^{1/\ell}) \left[(\ell-1) \frac{1}{\ell} + \frac{1}{\ell} - 1 \right] = 0,
\end{aligned}$$

because we used the hypothesis (ϕ_3) to obtain the last inequality. That is, $\Phi(t^{1/\ell})$ for $t > 0$ is a convex function.

So, it follows from (2.1) and of the convexity of $t \mapsto \Phi(t^{1/\ell})$ for $t > 0$, that

$$\begin{aligned}
J(\tau w_1 + (1-\tau)w_2) &= \int_{\Omega} \Phi(|\nabla z_3|) dx \\
&\leq \int_{\Omega} \Phi((\tau|\nabla z_1|^\ell + (1-\tau)|\nabla z_2|^\ell)^{\frac{1}{\ell}}) dx \\
&\leq \tau J(w_1) + (1-\tau)J(w_2),
\end{aligned}$$

showing that J is a convex functional.

Now, if we assumed that $\Omega_0 := \{x \in \Omega \mid u_1(x) > u_2(x)\}$ has positive Lebesgue measure, then $\varphi_i = (u_1^\ell - u_2^\ell)^+/u_i^{\ell-1}$, $i = 1, 2$ would be non-null admissible test functions, because $\varphi_1 = \varphi_2 = 0$ in Ω_0^c ,

$$|\nabla \varphi_i| \leq \ell \|u_j/u_i\|_\infty^{\ell-1} |\nabla \varphi_j| + [1 + (\ell-1)\|u_j/u_i\|_\infty^\ell] |\nabla \varphi_i|,$$

$u_i/u_j \in L^\infty(\Omega_0)$ for $i \neq j$, Lemma 5.1, and convexity of Φ .

So, it follows from the convexity of J , by using φ_i as test functions and the fact that $u_1, u_2 \in D(J)$, that

$$\begin{aligned}
0 &\leq \langle J'(u_1^\ell) - J'(u_2^\ell), u_1^\ell - u_2^\ell \rangle \\
&= \int_{\Omega} \phi(|\nabla u_1|) \nabla u_1 \nabla \left(\frac{u_1^\ell - u_2^\ell}{u_1^{\ell-1}} \right) - \phi(|\nabla u_2|) \nabla u_2 \nabla \left(\frac{u_1^\ell - u_2^\ell}{u_2^{\ell-1}} \right) dx. \tag{2.2}
\end{aligned}$$

Since u_1 is a subsolution and u_2 is a supersolution of problem (1.2), it follows from (2.2) and the fact that $t \mapsto f(x, t)/t^{\ell-1}$ is decreasing that

$$\begin{aligned}
0 &\leq \int_{\Omega} [\phi(|\nabla u_1|) \nabla u_1 \nabla \varphi_1 - \phi(|\nabla u_2|) \nabla u_2 \nabla \varphi_2] dx \\
&= \int_{\Omega} \left[\phi(|\nabla u_1|) \nabla u_1 \nabla \left(\frac{u_1^\ell - u_2^\ell}{u_1^{\ell-1}} \right) - \phi(|\nabla u_2|) \nabla u_2 \nabla \left(\frac{u_1^\ell - u_2^\ell}{u_2^{\ell-1}} \right) \right] dx \\
&\leq \int_{\Omega_0} (f(x, u_1) \varphi_1 - f(x, u_2) \varphi_2) dx \\
&= \int_{\Omega_0} \left(\frac{f(x, u_1)}{u_1^{\ell-1}} - \frac{f(x, u_2)}{u_2^{\ell-1}} \right) (u_1^\ell - u_2^\ell) dx < 0,
\end{aligned}$$

but this is impossible, that is, Ω_0 has null Lebesgue measure. This ends our proof. \square

3 Problem (1.1) regularized

Let us regularize Problem (1.1) by summing $\varepsilon > 0$ in singular term, and adding $\delta \geq 0$ on non-singular term. The last one is to ease the application of Theorem 1.2 in the proof of Theorem 1.1.

Let $\epsilon \in (0, 1)$ and $\delta \geq 0$. So, we associate with (1.1) the “regularized” problem

$$\begin{cases} -\Delta_{\Phi} u = \frac{\hat{a}_1(x)}{(u+\epsilon)^{\alpha_1}(v+\epsilon)^{\beta_1}} + \hat{b}_1(x)(u+\delta)^{\gamma_1}(v+\delta)^{\sigma_1}, & \text{in } \Omega, \\ -\Delta_{\Phi} v = \frac{\hat{a}_2(x)}{(u+\epsilon)^{\beta_2}(v+\epsilon)^{\alpha_2}} + \hat{b}_2(x)(u+\delta)^{\sigma_2}(v+\delta)^{\gamma_2} & \text{in } \Omega, \\ u, v \geq 0 \text{ in } \Omega, \quad u = v = 0 \text{ on } \partial\Omega. \end{cases} \quad (3.1)$$

So, we have.

Theorem 3.1 *Assume $(\phi_1) - (\phi_3)$ and $0 \leq \hat{a}_i, \hat{b}_i \in L^\infty(\Omega)$ hold with $\hat{a}_i + \hat{b}_i \neq 0$. Suppose in addition that $\alpha_i, \beta_i, \gamma_i, \sigma_i \geq 0$ with $\gamma_i + \sigma_i < \ell - 1$. Then there exists a weak solution $(u^1, u^2) = (u_{\varepsilon, \delta}^1, u_{\varepsilon, \delta}^2) \in W_0^{1, \Phi}(\Omega) \times W_0^{1, \Phi}(\Omega)$ of Problem (3.1) for each $\varepsilon > 0$ and $\delta \geq 0$ given. Beside this, $u^1, u^2 \neq 0$.*

Proof: Consider the vector space $E := W_0^{1, \Phi}(\Omega) \times W_0^{1, \Phi}(\Omega)$ endowed with the norm $\|(u, v)\| := \|u\| + \|v\|$. So, $(E, \|\cdot\|)$ is a reflexive Banach space. (We refer the reader to Section 5 for some basic facts on Orlicz-Sobolev spaces as well as references).

Consider the mapping $A = A_{\varepsilon, \delta} : E \times E \longrightarrow \mathbb{R}$ defined by

$$\begin{aligned} A(u_1, u_2, \varphi, \psi) &= \int_{\Omega} [\phi(|\nabla u_1|) \nabla u_1 \nabla \varphi + \phi(|\nabla u_2|) \nabla u_2 \nabla \psi] dx \\ &- \int_{\Omega} \frac{\hat{a}_1(x) \varphi}{(|u_1| + \epsilon)^{\alpha_1} (|u_2| + \epsilon)^{\beta_1}} + \frac{\hat{a}_2(x) \psi}{(|u_1| + \epsilon)^{\beta_2} (|u_2| + \epsilon)^{\alpha_2}} dx \\ &- \int_{\Omega} \hat{b}_1(x) (u_1^+ + \delta)^{\gamma_1} (u_2^+ + \delta)^{\sigma_1} \varphi dx \\ &- \int_{\Omega} \hat{b}_2(x) (u_1^+ + \delta)^{\sigma_2} (u_2^+ + \delta)^{\gamma_2} \psi dx. \end{aligned} \quad (3.2)$$

So, we have.

Proposition 3.1 *The functional A in (3.2) is well-defined, linear and satisfies:*

$$A(u_1, u_2, \cdot, \cdot) \in (E \times E)' = E' \times E' \text{ for each } (u_1, u_2) \in E.$$

Proof: Given $(u_1, u_2) \in E$, let $A_1 = A_{1, \varepsilon, \delta}$ be given by

$$\begin{aligned} A_1(u_1, u_2, \varphi) &:= \int_{\Omega} \phi(|\nabla u_1|) \nabla u_1 \nabla \varphi - \frac{\hat{a}_1(x) \varphi}{(|u_1| + \epsilon)^{\alpha_1} (|u_2| + \epsilon)^{\beta_1}} dx \\ &- \int_{\Omega} \hat{b}_1(x) (u_1^+ + \delta)^{\gamma_1} (u_2^+ + \delta)^{\sigma_1} \varphi dx, \end{aligned}$$

for $\varphi \in W_0^{1,\Phi}(\Omega)$ and $A_2 = A_{2,\varepsilon,\delta}$ defined by

$$\begin{aligned} A_2(u_1, u_2, \psi) &:= \int_{\Omega} [\phi(|\nabla u_2|) \nabla u_2 \nabla \psi - \frac{\hat{a}_2(x) \psi}{(|u_1| + \epsilon)^{\beta_2} (|u_2| + \epsilon)^{\alpha_2}} dx \\ &\quad - \int_{\Omega} \hat{b}_2(x) (u_1^+ + \delta)^{\sigma_2} (u_2^+ + \delta)^{\gamma_2} \psi dx, \end{aligned}$$

for $\psi \in W_0^{1,\Phi}(\Omega)$. That is, we are rewriting A as $A := (A_1, A_2)$.

Claim: $A_1(u_1, u_2, \varphi)$ is well-defined. It follows from Hölder's inequality, the embedding $W_0^{1,\Phi}(\Omega) \hookrightarrow L_{\Phi}(\Omega)$, the inequality $\tilde{\Phi}(t\phi(t)) \leq \Phi(2t)$ and the fact that $\Phi \in \Delta_2$, that

$$\begin{aligned} |A_1(u_1, u_2, \varphi)| &\leq \int_{\Omega} \phi(|\nabla u_1|) |\nabla u_1| |\nabla \varphi| + \frac{\hat{a}_1(x) |\varphi|}{\epsilon^{\alpha_1 + \beta_1}} dx \\ &\quad + \int_{\Omega} \hat{b}_1(x) (u_1^+ + \delta)^{\gamma_1} (u_2^+ + \delta)^{\sigma_1} |\varphi| dx \\ &\leq 2 \|\phi(|\nabla u_1|) |\nabla u_1|\|_{\tilde{\Phi}} \|\varphi\| + \frac{2}{\epsilon^{\alpha_1 + \beta_1}} \|\hat{a}_1\|_{\infty} \|\varphi\|_{\Phi} \\ &\quad + 2 \|\hat{b}_1(|u_1| + \delta)^{\gamma_1} (|u_2| + \delta)^{\sigma_1}\|_{\tilde{\Phi}} \|\varphi\|_{\Phi} \\ &\leq 2 \|\phi(|\nabla u_1|) |\nabla u_1|\|_{\tilde{\Phi}} \|\varphi\| + \frac{C}{\epsilon^{\alpha_1 + \beta_1}} \|\hat{a}_1\|_{\infty} \|\varphi\| \\ &\quad + C \|\hat{b}_1(|u_1| + \delta)^{\gamma_1} (|u_2| + \delta)^{\sigma_1}\|_{\tilde{\Phi}} \|\varphi\|. \end{aligned}$$

To end the proof of the Claim, it remains to show that

$$\|\hat{b}_1(|u_1| + \delta)^{\gamma_1} (|u_2| + \delta)^{\sigma_1}\|_{\tilde{\Phi}} < \infty.$$

Indeed, since $L_{\Phi}(\Omega) \hookrightarrow L^{(\gamma_1 + \sigma_1)\ell/(\ell-1)}(\Omega)$, because $L_{\Phi}(\Omega) \hookrightarrow L^{\ell}(\Omega)$ and $\gamma_1 + \sigma_1 \in (0, \ell - 1)$, we obtain by Lemma 5.2, that

$$\begin{aligned} &\int_{\Omega} \tilde{\Phi}(\hat{b}_1(x) (|u_1| + \delta)^{\gamma_1} (|u_2| + \delta)^{\sigma_1}) \\ &\leq \max\{\|\hat{b}_1\|_{\infty}^{\frac{\ell}{\ell-1}}, \|\hat{b}_1\|_{\infty}^{\frac{m}{m-1}}\} \int_{\Omega} \tilde{\Phi}((|u_1| + |u_2| + \delta + 1)^{\gamma_1 + \sigma_1}) dx \\ &\leq C \int_{\Omega} (|u_1| + |u_2| + \delta + 1)^{\frac{(\gamma_1 + \sigma_1)\ell}{\ell-1}} dx \\ &\leq C \| |u_1| + |u_2| + \delta + 1 \|_{\tilde{\Phi}}^{\frac{(\gamma_1 + \sigma_1)\ell}{\ell-1}} < \infty, \end{aligned} \tag{3.3}$$

that is,

$$|A_1(u_1, u_2, \varphi)| \leq [2 \|\phi(|\nabla u_1|) |\nabla u_1|\|_{\tilde{\Phi}} + \frac{C_1}{\epsilon^{\alpha_1 + \beta_1}} \|\hat{a}_1\|_{\infty} + C_2 \|(u_1, u_2)\|_{\tilde{\Phi}}^{\frac{(\gamma_1 + \sigma_1)\ell}{\ell-1}} + C_{\delta}] \|\varphi\|. \tag{3.4}$$

In a similar way one shows that

$$|A_2(u_1, u_2, \psi)| \leq [2 \|\phi(|\nabla u_2|) |\nabla u_2|\|_{\tilde{\Phi}} + \frac{D_1}{\epsilon^{\alpha_2 + \beta_2}} \|\hat{a}_2\|_{\infty} + D_2 \|(u_1, u_2)\|_{\tilde{\Phi}}^{\frac{(\gamma_2 + \sigma_2)\ell}{\ell-1}} + D_{\delta}] \|\psi\|, \tag{3.5}$$

for each $\varepsilon > 0$ and $\delta \geq 0$ given, where $C_i, D_i > 0$ and $C_{\delta}, D_{\delta} \geq 0$ with $C_{\delta} = D_{\delta} = 0$ if $\delta = 0$. The linearity is clear. These end the proof. \square

Proposition 3.2 *There is an only operator $T = T_{\varepsilon, \delta} : E \longrightarrow E'$ such that $\langle T(u_1, u_2), (\varphi, \psi) \rangle = A(u_1, u_2, \varphi, \psi)$ for all $(u_1, u_2), (\varphi, \psi) \in E$.*

Proof: Let $(u_1, u_2) \in E$. Of course, it there is at most one such T . In addition, by (3.4) and (3.5), we have

$$\begin{aligned} \|T(u_1, u_2)\|_{E'} &\leq 2\|\phi(|\nabla u_1|)|\nabla u_1|\|_{\tilde{\Phi}}\|\varphi\| + \frac{C_1}{\varepsilon^{\alpha_1+\beta_1}}\|\hat{a}_1\|_{\infty}\|\varphi\| \\ &+ C_2\|(u_1, u_2)\|^{\frac{(\gamma_1+\sigma_1)\ell}{\ell-1}}\|\varphi\| + C_{\delta}\|\varphi\|. \\ &+ 2\|\phi(|\nabla u_2|)|\nabla u_2|\|_{\tilde{\Phi}}\|\psi\| + \frac{D_1}{\varepsilon^{\alpha_2+\beta_2}}\|\hat{a}_2\|_{\infty}\|\psi\| \\ &+ D_2\|(u_1, u_2)\|^{\frac{(\gamma_2+\sigma_2)\ell}{\ell-1}}\|\varphi\| + D_{\delta}\|\psi\|, \end{aligned}$$

showing that $T(u_1, u_2) \in E'$. □

Our next aim is to show that there exist $(u^1, u^2) \in E \setminus \{0\}$ such that $T(u^1, u^2) = 0$. In fact, we will have that this (u^1, u^2) will be a non-negative weak solution of the system (3.1).

Now, since $a_i + b_i \neq 0$, we can take $(\omega_1, \omega_2) \in E$ such that

$$(\hat{a}_i + \hat{b}_i)\omega_i \neq 0 \text{ and } (\hat{a}_i + \hat{b}_i)\omega_i \in L^1(\Omega), \quad i = 1, 2. \quad (3.6)$$

From now on, let us consider the below set of linear subspaces of $W_0^{1,\Phi}(\Omega)$, that is,

$$\mathcal{A} = \left\{ F \subset W_0^{1,\Phi}(\Omega) \mid F \text{ is a linear subspace; } \omega_i \in F \text{ and } \dim F < \infty \right\}. \quad (3.7)$$

preordered by set inclusion.

Take a such $F \in \mathcal{A}$. Let $\beta = \{e_1, e_2, \dots, e_s\}$ be a linear basis of F , where $s := \dim F$ is denoting the dimension of F . So, there exist an unique $\xi^i = (\xi_1^i, \xi_2^i, \dots, \xi_s^i)$ such that

$$(u, v) = \sum_{j=1}^s (\xi_j^1 e_j, \xi_j^2 e_j),$$

for each $u, v \in F$ given.

Consider the isometric embedding

$$I_F : (F \times F, \|\cdot\|) \longrightarrow (E, \|\cdot\|) \text{ defined by } I_F(u_1, u_2) = (u_1, u_2),$$

and

$$T_F := I'_F \circ T \circ I_F : F \times F \longrightarrow F' \times F',$$

where I'_F is the adjoint of I_F . So, we have

$$\begin{aligned} \langle T_F(u_1, u_2), (\psi, \varphi) \rangle &= \langle I'_F \circ T \circ I_F(u_1, u_2), (\psi, \varphi) \rangle = \langle T \circ I_F(u_1, u_2), I_F(\psi, \varphi) \rangle \\ &= \langle T(u_1, u_2), (\psi, \varphi) \rangle \text{ for all } (u_1, u_2), (\psi, \varphi) \in F \times F. \end{aligned}$$

Thus

$$\begin{aligned}
\langle T_F(u_1, u_2), (\varphi, \psi) \rangle &= \int_{\Omega} [\phi(|\nabla u_1|) \nabla u_1 \nabla \varphi + \phi(|\nabla u_2|) \nabla u_2 \nabla \psi] dx \\
&- \int_{\Omega} \frac{\hat{a}_1(x) \varphi}{(|u_1| + \epsilon)^{\alpha_1} (|u_2| + \epsilon)^{\beta_1}} + \frac{\hat{a}_2(x) \psi}{(|u_1| + \epsilon)^{\beta_2} (|u_2| + \epsilon)^{\alpha_2}} dx \\
&- \int_{\Omega} \hat{b}_1(x) (u_1^+ + \delta)^{\gamma_1} (u_2^+ + \delta)^{\sigma_1} \varphi dx \\
&- \int_{\Omega} \hat{b}_2(x) (u_1^+ + \delta)^{\sigma_2} (u_2^+ + \delta)^{\gamma_2} \psi dx, \text{ for all } (u_1, u_2), (\psi, \varphi) \in F \times F
\end{aligned} \tag{3.8}$$

In this context, we have.

Proposition 3.3 *The operator T_F is continuous.*

Proof: To this end, we set $T_F = (T_1, T_2)$, where

$$\begin{aligned}
\langle T_1(u_1, u_2), \varphi \rangle &:= \int_{\Omega} \phi(|\nabla u_1|) \nabla u_1 \nabla \varphi - \frac{\hat{a}_1(x) \varphi}{(|u_1| + \epsilon)^{\alpha_1} (|u_2| + \epsilon)^{\beta_1}} dx \\
&- \int_{\Omega} \hat{b}_1(x) (u_1^+ + \delta)^{\gamma_1} (u_2^+ + \delta)^{\sigma_1} \varphi dx, \\
\langle T_2(u_1, u_2), \psi \rangle &:= \int_{\Omega} [\phi(|\nabla u_2|) \nabla u_2 \nabla \psi - \frac{\hat{a}_2(x) \psi}{(|u_1| + \epsilon)^{\beta_2} (|u_2| + \epsilon)^{\alpha_2}} dx \\
&- \int_{\Omega} \hat{b}_2(x) (u_1^+ + \delta)^{\sigma_2} (u_2^+ + \delta)^{\gamma_2} \psi dx,
\end{aligned}$$

for all $(u_1, u_2) \in F \times F$ and $\varphi, \psi \in F$, and we note that the operator $-\Delta_{\Phi}$ is continuous (see [12, Lemma 3.1]). Therefore, it just remains to show that

$$T_i - (-\Delta_{\Phi})|_F, \quad i = 1, 2$$

are continuous.

To show these, let $(u_{1,n}, u_{2,n}) \subseteq F \times F$ such that $(u_{1,n}, u_{2,n}) \rightarrow (u_1, u_2)$ in $F \times F$. So, passing to a subsequence if necessary, using Lemma 5.3 and the embedding $L_{\Phi}(\Omega) \hookrightarrow L^{\ell}(\Omega)$, we have

- (1) $u_{i,n} \rightarrow u_i$ a.e. in Ω , $i = 1, 2$;
- (2) there exist $h_i \in L^{\ell}(\Omega)$ such that $|u_{i,n}| \leq h_i$, $i = 1, 2$.

So,

$$\begin{aligned}
\frac{\hat{a}_1(x) \varphi}{(|u_{1,n}| + \epsilon)^{\alpha_1} (|u_{2,n}| + \epsilon)^{\beta_1}} &\xrightarrow{\text{a.e.}} \frac{\hat{a}_1(x) \varphi}{(|u_1| + \epsilon)^{\alpha_1} (|u_2| + \epsilon)^{\beta_1}}, \\
\hat{b}_1(x) (u_{1,n}^+ + \delta)^{\gamma_1} (u_{2,n}^+ + \delta)^{\sigma_1} \varphi &\xrightarrow{\text{a.e.}} \hat{b}_1(x) (u_1^+ + \delta)^{\gamma_1} (u_2^+ + \delta)^{\sigma_1} \varphi,
\end{aligned}$$

and, by using the facts that $\tilde{\Phi}$ is convex and it satisfies Δ_2 , we get

$$\tilde{\Phi} \left(\left| \frac{\hat{a}_1(x)}{(|u_{1,n}| + \epsilon)^{\alpha_1} (|u_{2,n}| + \epsilon)^{\beta_1}} - \frac{\hat{a}_1(x)}{(|u_1| + \epsilon)^{\alpha_1} (|u_2| + \epsilon)^{\beta_1}} \right| \right) \leq \tilde{\Phi} \left(\frac{2|\hat{a}_1|_{\infty}}{\epsilon^{\alpha_1 + \beta_1}} \right).$$

So, by Lebesgue's Theorem, we have

$$\int_{\Omega} \tilde{\Phi} \left(\left| \frac{\hat{a}_1(x)}{(|u_{1,n}| + \epsilon)^{\alpha_1}(|u_{2,n}| + \epsilon)^{\beta_1}} - \frac{\hat{a}_1(x)}{(|u_1| + \epsilon)^{\alpha_1}(|u_2| + \epsilon)^{\beta_1}} \right| \right) dx \rightarrow 0$$

or, in an equivalent way,

$$\left\| \frac{\hat{a}_1}{(|u_{1,n}| + \epsilon)^{\alpha_1}(|u_{2,n}| + \epsilon)^{\beta_1}} - \frac{\hat{a}_1}{(|u_1| + \epsilon)^{\alpha_1}(|u_2| + \epsilon)^{\beta_1}} \right\|_{\tilde{\Phi}} \rightarrow 0, \quad (3.9)$$

because $\tilde{\Phi} \in \Delta_2$. That is, by the Hölder inequality and (3.9), we obtain

$$\int_{\Omega} \left(\frac{\hat{a}_1(x)}{(|u_{1,n}| + \epsilon)^{\alpha_1}(|u_{2,n}| + \epsilon)^{\beta_1}} - \frac{\hat{a}_1(x)}{(|u_1| + \epsilon)^{\alpha_1}(|u_2| + \epsilon)^{\beta_1}} \right) v dx \rightarrow 0,$$

for each $v \in W_0^{1,\Phi}(\Omega)$.

By similar arguments to the above ones, we have

$$\begin{aligned} & \tilde{\Phi} \left(\hat{b}_1(x) |(u_{1,n}^+ + \delta)^{\gamma_1} (u_{2,n}^+ + \delta)^{\sigma_1} - (u_1^+ + \delta)^{\gamma_1} (u_2^+ + \delta)^{\sigma_1}| \right) \\ & \leq \tilde{\Phi} \left(2|\hat{b}_1|_{\infty} \frac{(h_1 + \delta)^{\gamma_1} (h_2 + \delta)^{\sigma_1} + (u_1^+ + \delta)^{\gamma_1} (u_2^+ + \delta)^{\sigma_1}}{2} \right) \\ & \leq C \left(\tilde{\Phi}((h_1 + h_2 + \delta)^{\gamma_1 + \sigma_1}) + \tilde{\Phi}((u_1^+ + u_2^+ + \delta)^{\gamma_1 + \sigma_1}) \right) \\ & \leq C \left((h_1 + h_2 + \delta)^{(\gamma_1 + \sigma_1)\ell/(\ell-1)} + (u_1^+ + u_2^+ + \delta)^{(\gamma_1 + \sigma_1)\ell/(\ell-1)} + C_{\delta} \right) \in L^1(\Omega), \end{aligned}$$

where $C_{\delta} \geq 0$. So, we have

$$\int_{\Omega} \hat{b}_1(x) [(u_{1,n}^+ + \delta)^{\gamma_1} (u_{2,n}^+ + \delta)^{\sigma_1} - (u_1^+ + \delta)^{\gamma_1} (u_2^+ + \delta)^{\sigma_1}] v dx \rightarrow 0.$$

These show that $T_1 - (-\Delta_{\Phi})|_F$ is continuous. In a similar way one shows that $T_2 - (-\Delta_{\Phi})|_F$ is continuous as well. Therefore T_F is continuous, ending the proof. \square

We are going to use the proposition below, which is a consequence of Brouwer's Fixed Point Theorem, see e.g. Lions [24].

Proposition 3.4 *Suppose that $S : \mathbb{R}^m \rightarrow \mathbb{R}^m$ is a continuous function such that $\langle S(\eta), \eta \rangle > 0$ on $|\eta| = r$, where $\langle \cdot, \cdot \rangle$ is the usual inner product in \mathbb{R}^m and $|\cdot|$ is its corresponding norm. Then, there exists $\eta_0 \in \overline{B}_r(0)$ such that $S(\eta_0) = 0$.*

To apply this proposition, we have to reduce our operator T_F to a finite dimensional space. To do this, define $S_F := i' \circ T_F \circ i : \mathbb{R}^s \times \mathbb{R}^s \rightarrow \mathbb{R}^s \times \mathbb{R}^s$, where $i = i_F : (\mathbb{R}^s \times \mathbb{R}^s, |\cdot|) \rightarrow (F \times F, \|\cdot\|)$, given by $i(\xi^1, \xi^2) = (u, v)$, is an isometry with the norm in $\mathbb{R}^s \times \mathbb{R}^s$ given by $|(\xi^1, \xi^2)| := \|(u, v)\| := \|u\| + \|v\|$, and i' is its adjoint operator.

Proposition 3.5 *The operator S_F admits one zero $(\xi^1, \xi^2) = (\xi_F^{1,\varepsilon,\delta}, \xi_F^{2,\varepsilon,\delta})$ in $\mathbb{R}^s \times \mathbb{R}^s$ with $\xi^1, \xi^2 \neq 0$. Moreover, the corresponding vector $(u_1, u_2) = (u_F^{1,\varepsilon,\delta}, u_F^{2,\varepsilon,\delta}) = i(\xi^1, \xi^2) \in F \times F$ satisfies $T_F(u_1, u_2) = 0$ and $u_1, u_2 \neq 0$.*

Proof. Given $(\xi^1, \xi^2) \in \mathbb{R}^s \times \mathbb{R}^s$, denote the by $(u_1, u_2) \in F \times F$ the only image vector of (ξ^1, ξ^2) by the isometry i . So, by using (ϕ_3) , we have

$$\begin{aligned}
(S_F(\xi^1, \xi^2), (\xi^1, \xi^2)) &= (i^* \circ T_F \circ i(\xi^1, \xi^2), (\xi^1, \xi^2)) = (T_F(u_1, u_2), (u_1, u_2)) \\
&\geq \int_{\Omega} [\phi(|\nabla u_1|)|\nabla u_1|^2 + \phi(|\nabla u_2|)|\nabla u_2|^2] dx \\
&- \int_{\Omega} \frac{\hat{a}_1(x)}{\epsilon^{\alpha_1+\beta_1}} |u_1| + \frac{\hat{a}_2(x)}{\epsilon^{\alpha_2+\beta_2}} |u_2| dx \\
&- \int_{\Omega} \hat{b}_1(x)(u_1^+ + \delta)^{\gamma_1+1} (u_2^+ + \delta)^{\sigma_1} dx \\
&- \int_{\Omega} \hat{b}_2(x)(u_1^+ + 1)^{\sigma_2} (u_2^+ + 1)^{\gamma_2+1} dx.
\end{aligned} \tag{3.10}$$

Now, by using (Φ_3) and following similar arguments as done in (3.3), we obtain

$$\begin{aligned}
(S_F(\xi^1, \xi^2), (\xi^1, \xi^2)) &\geq \ell \min\{\|u_1\|^\ell, \|u_1\|^m\} + \ell \min\{\|u_2\|^\ell, \|u_2\|^m\} \\
&- C_1^\epsilon \|(u_1, u_2)\| - C_2^\delta \|(u_1, u_2)\|^{\gamma_1+\sigma_1+1} \\
&- C_3^\delta \|(u_1, u_2)\|^{\gamma_2+\sigma_2+1} - C_4^\delta,
\end{aligned}$$

where $C_1^\epsilon, C_2^\delta, C_3^\delta > 0$, and $C_4^\delta \geq 0$ are real constants.

So, by setting $r_0 := \|(u_1, u_2)\| = r_1 + r_2 \geq 2$, we have $r_1 := \|u_1\| \geq 1$ or $r_2 := \|u_2\| \geq 1$, and

$$r_0^\ell = (r_1 + r_2)^\ell \leq 2^\ell \min\{r_1^\ell, r_1^m\} + 2^\ell \min\{r_2^\ell, r_2^m\},$$

that is,

$$(S_F(\xi_1, \xi_2), (\xi_1, \xi_2)) \geq \frac{\ell}{2^\ell} r_0^\ell - C_1^\epsilon r_0 - C_2^\delta r_0^{\gamma_1+\sigma_1+1} - C_3^\delta r_0^{\gamma_2+\sigma_2+1} - C_4^\delta.$$

Since, $\gamma_j + \sigma_j \in (0, \ell - 1)$, we can choose an $r_0 = r_0^{\epsilon, \delta} > 0$ such that

$$(S_F(\xi_1, \xi_2), (\xi_1, \xi_2)) > 0 \text{ for all } \|(\xi_1, \xi_2)\| = r_0$$

which implies, by applying Proposition 3.4, that there is a $(\xi^1, \xi^2) = (\xi_F^{1, \epsilon, \delta}, \xi_F^{2, \epsilon, \delta}) \in \overline{B}_{r_0}(0, 0) \subset \mathbb{R}^s \times \mathbb{R}^s$ such that $S_F(\xi^1, \xi^2) = (0, 0)$. Let $(u_1, u_2) = (u_F^{1, \epsilon, \delta}, u_F^{2, \epsilon, \delta}) = i(\xi_F^{1, \epsilon, \delta}, \xi_F^{2, \epsilon, \delta})$. Then

$$\langle T_F(u_1, u_2), (\psi, \varphi) \rangle = (S_F(\xi^1, \xi^2), (\eta_1, \eta_2)) = 0,$$

for all $(\psi, \varphi) = i(\eta_1, \eta_2) \in F \times F$, that is,

$$\begin{aligned}
\int_{\Omega} \phi(|\nabla u_1|) \nabla u_1 \nabla \psi &= \int_{\Omega} \frac{\hat{a}_1(x) \psi}{(|u_1| + \epsilon)^{\alpha_1} (|u_2| + \epsilon)^{\beta_1}} dx \\
&+ \int_{\Omega} \hat{b}_1(x) (u_1^+ + \delta)^{\gamma_1} (u_2^+ + \delta)^{\sigma_1} \psi dx, \quad \psi \in F
\end{aligned}$$

and

$$\begin{aligned}
\int_{\Omega} [\phi(|\nabla u_2|) \nabla u_2 \nabla \varphi] &= \int_{\Omega} \frac{\hat{a}_2(x) \varphi}{(|u_1| + \epsilon)^{\beta_2} (|u_2| + \epsilon)^{\alpha_2}} dx \\
&+ \int_{\Omega} \hat{b}_2(x) (u_1^+ + \delta)^{\sigma_2} (u_2^+ + \delta)^{\gamma_2} \varphi dx, \quad \varphi \in F.
\end{aligned}$$

In particular, we have that $u_1 \not\equiv 0$, because otherwise

$$0 = \int_{\Omega} \frac{\hat{a}_1(x)\varphi}{\epsilon^{\beta_1}(|u_2| + \epsilon)^{\alpha_1}} dx + \int_{\Omega} \hat{b}_1(x)\delta^{\sigma_1}(u_2^+ + \delta)^{\gamma_1}\varphi dx, \quad \varphi \in F.$$

contradicting (3.6). In a similar way, we show that $u_2 \neq 0$ as well. \square

The next result is an immediate consequence of the proof of Proposition 3.5.

Corollary 3.1 *Assume $\alpha_1, \alpha_2 \geq 0$ and $0 \leq \gamma_i + \sigma_i < \ell - 1$. Then there are an $r_0 = r_0^{\varepsilon, \delta} > 0$ and $0 \neq u_1, u_2 \in F$, for each $F \in \mathcal{A}$ given, such that $T_F(u_1, u_2) = (0, 0)$ and $\|(u_1, u_2)\| \leq r_0$ with r_0 does not depending on F . Besides this, r_0 does not depend on $\varepsilon > 0$ if $\beta_i = 0$ and $\alpha_i \in (0, 1]$.*

Proof. The first part of Corollary was just proved by above arguments. To show that r_0 does not depend on $\varepsilon > 0$, it is necessary just going back to (3.10) and redo the below estimate, say

$$\begin{aligned} \int_{\Omega} \frac{\hat{a}_i(x)}{(|u_i| + \varepsilon)^{\alpha_i}} u_i &\leq \int_{|u_i| \leq 1} \hat{a}_i |u_i|^{1-\alpha_i} + \int_{|u_i| > 1} \hat{a}_i |u_i|^{1-\alpha_i} \\ &\leq \|a_i\|_1 + \|a_i\|_{\infty} \|u_i\|. \end{aligned}$$

Now, following similar arguments like those that were used above, we show the existence of one $r_0 > 0$ independent of $\varepsilon > 0$. \square

After these, we are able to solve the equation $T(u_1, u_2) = 0$, where the operator T was given by Proposition 3.2. More, this zero of T will be a solution of (3.1). To solve $T(u_1, u_2) = 0$, we have inspired in an idea found in the work of Browder [4].

Lemma 3.1 *For each small $\epsilon > 0$, it there is a $(u^1, u^2) = (u_{\epsilon, \delta}^1, u_{\epsilon, \delta}^2) \in E$, with $u^1, u^2 \neq 0$, such that $T(u^1, u^2) = 0$.*

Proof. Let $F_0 \in \mathcal{A}$ and define

$$V_{F_0} = \{(u_1, u_2) \in F \times F \mid F \in \mathcal{A}, F_0 \subset F, T_F(u_1, u_2) = 0, \|(u_1, u_2)\| \leq r_0\},$$

where $r_0 > 0$ was defined at Corollary 3.1, and \mathcal{A} was defined in (3.7).

By Proposition 3.5 and Corollary 3.1, we have that $V_{F_0} \neq \emptyset$ and $\overline{V_{F_0}}^{\sigma} \subset B_{r_0}$, where $\overline{V_{F_0}}^{\sigma}$ is the weak closure of V_{F_0} and B_{r_0} is the closed ball. So, $\overline{V_{F_0}}^{\sigma}$ is weakly compact.

Claim. The family

$$\mathcal{B} := \{\overline{V_F}^{\sigma} \mid F \in \mathcal{A}\}$$

has the finite intersection property.

Indeed, consider the finite family $\{\overline{V_{F_1}}^{\sigma}, \overline{V_{F_2}}^{\sigma}, \dots, \overline{V_{F_p}}^{\sigma}\} \subset \mathcal{B}$ and let $F := \text{span}\{F_1, F_2, \dots, F_p\}$. So, by the definition of V_{F_i} , $u_F \in \overline{V_{F_i}}^{\sigma}$, $i = 1, 2, \dots, p$, and so

$$\bigcap_{i=1}^p \overline{V_{F_i}}^{\sigma} \neq \emptyset,$$

showing the **Claim**.

Since B_{r_0} is weakly compact and \mathcal{B} has the finite intersection property, it follows by [26, Thm. 26.9] that

$$W := \bigcap_{F \in \mathcal{A}} \overline{V}_F^\sigma \neq \emptyset.$$

Let $(u^1, u^2) = (u_{\epsilon, \delta}^1, u_{\epsilon, \delta}^2) \in W$. Then $T(u^1, u^2) = 0$, or equivalently,

$$\begin{aligned} \int_{\Omega} \phi(|\nabla u^1|) \nabla u^1 \nabla \psi dx &= \int_{\Omega} \frac{\hat{a}_1(x) \psi}{(|u^1| + \epsilon)^{\alpha_1} (|u^2| + \epsilon)^{\beta_1}} dx \\ &+ \int_{\Omega} \hat{b}_1(x) (u^{1+} + \delta)^{\gamma_1} (u^{2+} + \delta)^{\sigma_1} \psi dx \end{aligned} \quad (3.11)$$

and

$$\begin{aligned} \int_{\Omega} \phi(|\nabla u^2|) \nabla u^2 \nabla \varphi dx &= \int_{\Omega} \frac{\hat{a}_2(x) \varphi}{(|u^1| + \epsilon)^{\beta_2} (|u^2| + \epsilon)^{\alpha_2}} dx \\ &+ \int_{\Omega} \hat{b}_2(x) (u^{1+} + \delta)^{\sigma_2} (u^{2+} + \delta)^{\gamma_2} \varphi dx \end{aligned} \quad (3.12)$$

for all $\psi, \varphi \in W_0^{1, \Phi}(\Omega)$.

Indeed, take $F_0 = \text{span}\{\omega_1, \omega_2, \psi, \varphi, u^1, u^2\}$. So, $F_0 \in \mathcal{A}$ and by [10, Thm. 1.5] there is a sequence $(u_n^1, u_n^2) \subseteq V_{F_0}$ such that $(u_n^1, u_n^2) \rightharpoonup (u^1, u^2)$ in E . Since $(u_n^1, u_n^2) \in V_{F_0}$, it follows from its definition that $\|(u_n^1, u_n^2)\| \leq r_0$ and there exists a $F_n \in \mathcal{A}$, with $F_0 \subset F_n$, such that

$$\begin{aligned} \int_{\Omega} \phi(|\nabla u_n^1|) \nabla u_n^1 \nabla \eta_1 dx &= \int_{\Omega} \frac{\hat{a}_1(x) \eta_1}{(|u_n^1| + \epsilon)^{\alpha_1} (|u_n^2| + \epsilon)^{\beta_1}} dx \\ &+ \int_{\Omega} \hat{b}_1(x) (u_n^{1+} + \delta)^{\gamma_1} (u_n^{2+} + \delta)^{\sigma_1} \eta_1 dx, \end{aligned} \quad (3.13)$$

and

$$\begin{aligned} \int_{\Omega} \phi(|\nabla u_n^2|) \nabla u_n^2 \nabla \eta_2 dx &= \int_{\Omega} \frac{\hat{a}_2(x) \eta_2}{(|u_n^1| + \epsilon)^{\beta_2} (|u_n^2| + \epsilon)^{\alpha_2}} dx \\ &+ \int_{\Omega} \hat{b}_2(x) (u_n^{1+} + \delta)^{\sigma_2} (u_n^{2+} + \delta)^{\gamma_2} \eta_2 dx \end{aligned} \quad (3.14)$$

for all $\eta_1, \eta_2 \in F_n$.

Now, since $W_0^{1, \Phi}(\Omega) \xrightarrow{cpt} L_{\Phi}(\Omega) \hookrightarrow L^{\ell}(\Omega)$, we have that:

$$(1) \quad u_n^1 \rightarrow u^1 \text{ and } u_n^2 \rightarrow u^2 \text{ in } L_{\Phi}(\Omega) \text{ and } L^{\ell}(\Omega),$$

$$(2) \quad u_n^1 \rightarrow u^1 \text{ and } u_n^2 \rightarrow u^2 \text{ a.e. in } \Omega.$$

Set $\eta_i = u_n^i - u^i \in F_n$ for $i = 1, 2$. So, it follows from the definition and properties of \mathcal{A} that $\eta_i \in F_n$. Now, by using in η_1 as a test function in (3.13), we get

$$\begin{aligned}
\lim \langle (-\Delta_\Phi)(u_n^1), u_n^1 - u^1 \rangle &= \lim \int_\Omega \frac{\hat{a}_1(x)(u_n^1 - u^1)}{(|u_n^1| + \epsilon)^{\alpha_1}(|u_n^2| + \epsilon)^{\beta_1}} dx \\
&+ \lim \int_\Omega \hat{b}_1(x)(u_n^{1+} + \delta)^{\gamma_1}(u_n^{2+} + \delta)^{\sigma_1}(u_n^1 - u^1) dx \\
&\leq \lim \int_\Omega \frac{\hat{a}_1(x)}{\epsilon^{\alpha_1 + \beta_1}} |u_n^1 - u^1| dx. \\
&+ \lim \int_\Omega \hat{b}_1(x)(|u_n^{1+}| + |u_n^{2+}| + \delta)^{\gamma_1 + \sigma_1} |u_n^1 - u^1| dx.
\end{aligned} \tag{3.15}$$

Since $W_0^{1,\Phi}(\Omega) \xrightarrow{cpt} L^1(\Omega)$, it follows from (1) above, that

$$\int_\Omega \frac{\hat{a}_1(x)}{\epsilon^{\alpha_1 + \beta_1}} |u_n^1 - u^1| dx \leq \frac{|\hat{a}_1|_\infty}{\epsilon^{\alpha_1 + \beta_1}} \int_\Omega |u_n^1 - u^1| dx \rightarrow 0, \tag{3.16}$$

and by using $\gamma_1 + \sigma_1 < \ell - 1$, $W_0^{1,\Phi}(\Omega) \hookrightarrow L^\ell(\Omega)$ and (1) again, we obtain

$$\begin{aligned}
&\int_\Omega \hat{b}_1(x)(|u_n^{1+}| + |u_n^{2+}| + \delta)^{\gamma_1 + \sigma_1} |u_n^1 - u^1| dx \\
&\leq |\hat{b}_1|_\infty \left(\int_\Omega (|u_n^{1+}| + |u_n^{2+}| + \delta)^{\frac{(\gamma_1 + \sigma_1)\ell}{\ell-1}} \right)^{\frac{\ell-1}{\ell}} |u_n^1 - u^1|_\ell \\
&\leq |\hat{b}_1|_\infty \left(\int_\Omega (|u_n^{1+}| + |u_n^{2+}| + \delta)^\ell \right)^{\frac{\ell-1}{\ell}} |u_n^1 - u^1|_\ell.
\end{aligned} \tag{3.17}$$

So, as a consequence of (3.15), (3.16) and (3.17), we have

$$\lim \langle (-\Delta_\Phi)(u_n^1), u_n^1 - u^1 \rangle \leq 0,$$

which implies that $u_n^1 \rightarrow u^1$ in $W_0^{1,\Phi}(\Omega)$, because $(-\Delta_\Phi)$ is an operator of the type (S_+) (see [5, Prop. A.2]). By a similar argument one shows that $u_n^2 \rightarrow u^2$ in $W_0^{1,\Phi}(\Omega)$.

Verification of (3.11) and (3.12): Passing to a subsequence, if necessary, we have

- (1) $\nabla u_n^i \rightarrow \nabla u^i$ a.e in Ω ,
- (2) there exist $h_i \in L_\Phi(\Omega)$ such that $|\nabla u_n^i| \leq h_i$.

By the Young's inequality for N-functions, we have

$$\begin{aligned}
|\phi(|\nabla u_n^1|)|\nabla u_n^1 \nabla \varphi| &\leq \phi(|\nabla u_n^1|)|\nabla u_n^1| |\nabla \varphi| \leq \phi(h_1)h_1 |\nabla \varphi| \\
&\leq \tilde{\Phi}(\phi(h_1)h_1) + \Phi(|\nabla \varphi|) \\
&\leq \Phi(2h_1) + \Phi(|\nabla \varphi|) \in L^1(\Omega),
\end{aligned}$$

where we have used the hypothesis (ϕ_2) , that is, the function $t \mapsto t\phi(t)$ is increasing for $t \geq 0$.

Now, applying the Lebesgue's Theorem one finds that

$$\int_{\Omega} \phi(|\nabla u_n^1|) \nabla u_n^1 \nabla \psi dx \longrightarrow \int_{\Omega} \phi(|\nabla u^1|) \nabla u^1 \nabla \psi dx,$$

and arguing as above we also have

$$\int_{\Omega} \phi(|\nabla u_n^2|) \nabla u_n^2 \nabla \varphi dx \longrightarrow \int_{\Omega} \phi(|\nabla u^2|) \nabla u^2 \nabla \varphi dx.$$

Setting $\eta_1 = \psi$ and $\eta_2 = \varphi$ in (3.13) and (3.14), respectively, and passing to the limit, we obtain (3.11) and (3.12), that is, $T(u^1, u^2) = (0, 0)$. Finally, by similar arguments to those used in Proposition 3.5, we infer that $u^1, u^2 \not\equiv 0$ as well. \square

To finish the proof of Theorem 3.1, it just remains to show that $u^1, u^2 \geq 0$. Taking $-(u^1)^-$ as a test function in Equation (3.11), it follows from (Φ_3) that

$$\begin{aligned} \ell \int_{\Omega} \Phi(|\nabla u^{1-}|) dx &\leq \int_{\Omega} \phi(|\nabla u^{1-}|) |\nabla u^{1-}|^2 dx \\ &= - \int_{\Omega} \frac{\hat{a}_1(x) u^{1-}}{(u^{1-} + \epsilon)^{\alpha_1} (|u^2| + \epsilon)^{\beta_1}} dx \\ &\quad - \int_{\Omega} \hat{b}_1(x) (u^{1+} + \delta)^{\gamma_1} (u^{2+} + \delta)^{\sigma_1} u^{1-} dx \leq 0, \end{aligned}$$

that is, $u^{1-} \equiv 0$. In a similar way one shows that $u^{2-} \equiv 0$. This ends our proof. \square

4 Proof of Theorem 1.1 (final arguments).

In this section, we will consider $a_n^i = \max\{a_i, n\}$, and $b_n^i = \max\{b_i, n\}$ for $n \in \mathbb{N}$. So, $a_n^i, b_n^i \in L^\infty(\Omega)$ for all $n \in \mathbb{N}$ in accordance to apply Theorem 3.1.

Proof of (i): In this case, Problem (3.1) read as

$$\begin{cases} -\Delta_{\Phi} u = \frac{a_1^n(x)}{(u + \frac{1}{n})^{\alpha_1}} + b_1^n(x) u^{\gamma_1} v^{\sigma_1}, \\ -\Delta_{\Phi} v = \frac{a_2^n(x)}{(v + \frac{1}{n})^{\alpha_2}} + b_2^n(x) u^{\sigma_2} v^{\gamma_2}, \text{ in } \Omega, \\ u, v \geq 0 \text{ in } \Omega, \quad u = v = 0 \text{ on } \partial\Omega, \end{cases} \quad (4.1)$$

where $\delta = 0$, $\varepsilon = 1/n$, $n \in \mathbb{N}$. So, it follows from Theorem 3.1 that there exists an $(u_n, v_n) \in E$, with $u_n, v_n \neq 0$, solution of (4.1).

Claim. $u_n + 1/n \geq Cd(x)$ and $v_n + 1/n \geq Cd(x)$ for some $C > 0$.

Indeed, by taking $\hat{b}_i = 0$ and applying Theorem 3.1 with $\varepsilon = 1$, we obtain a $w_i \in W_0^{1,\Phi}(\Omega)$ solution of the problem

$$\begin{cases} -\Delta_{\Phi} w = \frac{a_i^1(x)}{(w + 1)^{\alpha_i}} \text{ in } \Omega, \\ w \geq 0 \text{ in } \Omega; \quad w = 0 \text{ on } \partial\Omega. \end{cases} \quad (4.2)$$

Besides this, it follows from [12, Lemma 3.3], that $w_i \in C^{1,\beta_i}(\overline{\Omega})$, for some $0 < \beta_i < 1$, and from [5, Prop. 5.2], it follows that $w_i > 0$.

Since the solution (u_n, v_n) of (4.1) satisfies

$$-\Delta_{\Phi} u_n \geq \frac{a_1^n(x)}{(u_n + 1/n)^{\alpha_1}} \geq \frac{a_1^1(x)}{(u_n + 1)^{\alpha_1}} \text{ in } \Omega,$$

it follows that w_1 and u_n are respectively sub e supersolutions of (4.2) with $i = 1$ such that

$$0 \leq \frac{w_1}{u_n + \frac{1}{n}} \leq nw_1 \in L^{\infty}(\Omega).$$

So, by Theorem 1.2, we obtain

$$u_n + \frac{1}{n} \geq w_1 > 0 \text{ a.e. in } \Omega. \quad (4.3)$$

Finally, by a classical argument, we can show that $w_1 \geq C_1 d$, for some $C_1 > 0$. In a similar way, we have

$$v_n(x) + \frac{1}{n} \geq w_2(x) \geq C_2 d(x) \text{ a.e. in } \Omega, \quad (4.4)$$

as well. This ends the proof of Claim.

Now, by using $u_n \in W_0^{1,\Phi}(\Omega)$ as a test function in the first equation in (3.1), the hypothesis (ϕ_3) ((ϕ_3) implies $(\phi_3)'$, see Remark 5.1), the arguments as in (3.3), and (4.3), we obtain

$$\begin{aligned} \ell \zeta_0(\|\nabla u_n\|_{\Phi}) &\leq \ell \int_{\Omega} \Phi(|\nabla u_n|) \leq \int_{\Omega} \phi(|\nabla u_n|) |\nabla u_n|^2 \\ &\leq \int_{\Omega} \frac{a_1(x)}{(u_n + 1/n)^{\alpha_1}} u_n dx + \int_{\Omega} b_1(x) u_n^{\gamma_1} v_n^{\sigma_1} u_n dx \\ &= \int_{\Omega} \frac{a}{d^{\alpha_1}} u_n dx + \int_{\Omega} b_1(v_n + u_n)^{\gamma_1 + \sigma_1 + 1} dx \\ &\leq \|a/d^{\alpha_1}\|_{\tilde{\Psi}} \|u_n\|_{\Psi} + C_2' \|b_1\|_{q_1} \|(u_1, u_2)\|^{\gamma_1 + \sigma_1 + 1} \\ &\leq C_1 \|u_n\| + C_2 \|(u_1, u_2)\|^{\gamma_1 + \sigma_1 + 1}, \end{aligned} \quad (4.5)$$

and, in a similar way, we obtain

$$\ell \zeta_0(\|\nabla v_n\|_{\Phi}) \leq D_1 \|v_n\| + D_2 \|(u_1, u_2)\|^{\gamma_2 + \sigma_2 + 1}. \quad (4.6)$$

So, by using either (4.5) or (4.6), we can show that $(u_n, v_n) \subset E$ is bounded if either $\|v_n\| \geq 1$ and $\|u_n\| \leq 1$ or $\|v_n\| \leq 1$ and $\|u_n\| \geq 1$. Now, assume that $\|u_n\|, \|v_n\| \geq 1$. So, by summing (4.5) and (4.6), it follows from Lemma 5.1, that

$$\|u_n\|^{\ell} + \|v_n\|^{\ell} \leq D_1' \|(u_n, v_n)\| + D_2' \|(u_1, u_2)\|^{\gamma_1 + \sigma_1 + 1} + D_3' \|(u_1, u_2)\|^{\gamma_2 + \sigma_2 + 1}.$$

for some $D_i' > 0$. Since, $\gamma_i + \sigma_i < \ell - 1$, we have that $(u_n, v_n) \subset E$ is bounded as well. Passing to a subsequence if necessary, we find that

$$(1) \quad u_n \rightharpoonup u \text{ and } v_n \rightharpoonup v \text{ in } W_0^{1,\Phi}(\Omega);$$

(2) $u_n \rightarrow u$ and $v_n \rightarrow v$ in $L_\Phi(\Omega)$;

(3) $u_n \rightarrow u$ e $v_n \rightarrow v$ a.e. in Ω ;

(4) there exist $\theta_i \in L_\Phi(\Omega)$ such that $0 < C_1 d \leq u_n + 1/n \leq \theta_1$ and $0 < C_2 d \leq v_n + 1/n \leq \theta_2$,

that is, by using (3), (4.3) and (4.4) we have that $u, v \geq Cd$ a.e. in Ω for some $C > 0$.

Next we will show that $(u_n, v_n) \rightarrow (u, v)$ in E . Indeed, by using (4) and $a/d^{\alpha_1} \in L_{\tilde{\Phi}}$, we obtain

$$\int_{\Omega} \frac{a_1^n(u_n - u)}{(u_n + \frac{1}{n})^{\alpha_1}} \leq \int_{\Omega} \frac{a_1}{d^{\alpha_1}} |u_n - u| \leq 2 \|a_1/d^{\alpha_1}\|_{\tilde{\Phi}} \|u_n - u\|_{\Phi}$$

and

$$|b_1(x) u_n^{\gamma_1} v_n^{\sigma_1} (u_n - u)| \leq 2b_1(x) \max\{\theta_1, \theta_2\}^{\gamma_1 + \sigma_1 + 1} \in L^1(\Omega),$$

because $b_1 \in L^{q_1}(\Omega)$ and $L^\Phi(\Omega) \hookrightarrow L^{\gamma_1 + \sigma_1 + 1}(\Omega)$. So, by applying Fatou's Lemma, it follows from (4.1) that

$$\limsup \langle (-\Delta_\Phi) u_n, u_n - u \rangle \leq 0,$$

that is, $u_n \rightarrow u$ in $W_0^{1,\Phi}(\Omega)$. Similarly one shows that $v_n \rightarrow v$ in $W_0^{1,\Phi}(\Omega)$. Passing to the limit in (4.1) we infer that (u, v) is a weak solution of (1.1). These end the proof of (i).

About Remark 1.1, it is necessary just to observe that r_0 given by Corollary 3.1 does not depend on n , that is, $(u_n, v_n) \subset E$ is already bounded. So, the remaining arguments are the same.

Proof of (ii): In this case, the problem (3.1) reduces to

$$\begin{cases} -\Delta_\Phi u = \frac{a_1^n(x)}{(u + 1/n)^{\alpha_1} (v + 1/n)^{\beta_1}} + b_1^n(x) (u + 1/n)^{\gamma_1} & \text{in } \Omega, \\ -\Delta_\Phi v = \frac{a_2^n(x)}{(u + 1/n)^{\beta_2} (v + 1/n)^{\alpha_2}} + b_2^n(x) (v + 1/n)^{\gamma_2} & \text{in } \Omega, \\ u, v \geq 0 & \text{in } \Omega, \quad u = v = 0 \text{ on } \partial\Omega, \end{cases} \quad (4.7)$$

by taking $\varepsilon = \delta = 1/n$ with $n \in \mathbb{N}$. So, it follows from Theorem 3.1 that there exists an $(u_n, v_n) \in E$, with $u_n, v_n \neq 0$, solution of (4.7).

Besides this, with similar arguments as those used in Case (i), we able to show that

$$u_n + 1/n \geq Cd, \quad v_n + 1/n \geq Cd, \quad \text{for some } C > 0.$$

In this case, we redo the above arguments using $w_i \in C^{1,\tau_i}(\overline{\Omega})$, for some $0 < \tau_i < 1$, as solution of the problem

$$\begin{cases} -\Delta_\Phi w = b_i^1(x) w^{\gamma_i} & \text{in } \Omega, \\ w > 0 & \text{in } \Omega, \quad w = 0 \text{ on } \partial\Omega, \end{cases}$$

in what the existence result is given by Theorem 3.1 with $\hat{a}_i = 0$, and $\delta = 0$. The regularity is guaranteed by [29, Corollary 3.1], and the positivity is given by [5, Prop. 5.2] again.

After this, in a similar way to those that we have done in case (i), we are able to show that $(u_n, v_n) \subset E$ is bounded as well. This ends the proof of Theorem 1.1 - (ii).

Proof of (iii): Under these conditions, the system (3.1) becomes

$$\begin{cases} -\Delta_{\Phi} u = \frac{a_1^n(x)}{(v + 1/n)^{\beta_1}} + b_1^n(x)(v + 1/n)^{\sigma_1}, \\ -\Delta_{\Phi} v = \frac{a_2^n(x)}{(u + 1/n)^{\beta_2}} + b_2^n(x)(u + 1/n)^{\sigma_2}, \text{ in } \Omega, \\ u, v \geq 0 \text{ in } \Omega, \quad u = v = 0 \text{ on } \partial\Omega. \end{cases} \quad (4.8)$$

with $\varepsilon = \delta = 1/n$ and $n \in \mathbb{N}$.

So, it follows from Theorem 3.1 that there exists a weak solution $(u_n, v_n) \in E$ to Problem (4.8).

Claim. there exists a constant $C > 0$ such that $u_n \geq Cd$ and $v_n \geq Cd$.

Indeed, let us set $h_i(x) := \min\{a_i(x), b_i(x)\}$, $x \in \Omega$, and

$$g_i := \min_{t \geq 0} \left[\frac{1}{t^{\beta_i} + 1} + t^{\sigma_i} \right].$$

So, we can infer from our hypotheses that

$$h_i(x) \geq 0, \quad h_i \not\equiv 0 \text{ and } g_i > 0.$$

Now, it follows from [12, Lemma 3.4] that there is a positive solution $w_i \in W_0^{1,\Phi}(\Omega)$ of

$$\begin{cases} -\Delta_{\Phi} w = h_i(x)g_i \text{ in } \Omega, \\ w \geq 0 \text{ in } \Omega, \quad w = 0 \text{ on } \partial\Omega. \end{cases} \quad (4.9)$$

Again, it follows from [29, Corolary [3.1] that $w_i \in C^{1,\tau_i}(\overline{\Omega})$, for some $0 < \tau_i < 1$, and by [5, Prop. 5.2], we have $w_i > 0$, thais is, there exists a $C_i > 0$ such that $w_i \geq C_i d$.

So, it follows from (4.8) and (4.9), that

$$\begin{cases} -\Delta_{\Phi} u_n \geq h_1(x)g_1 = -\Delta_{\Phi} w_1 \text{ in } \Omega, \\ -\Delta_{\Phi} v_n \geq h_2(x)g_2 = -\Delta_{\Phi} w_2 \text{ in } \Omega, \\ u_n = v_n = w_1 = w_2 = 0 \text{ on } \partial\Omega, \end{cases}$$

that implies, by applying [29, Lemma 4.1], that $u_n \geq w_1 \geq C_1 d$ and $v_n \geq w_2 \geq C_2 d$, for some $C_1, C_2 > 0$.

Finally, arguing as in the proof of Theorem 1.1-(i), we are able to show that $(u_n, v_n) \subset E$ is bounded as well. These end the proof of Theorem 1.1 - (iii) and the proof of Theorem 1.1. \square

5 Appendix

In this section we present for, the reader's convenience, several results used in this paper. We begin referring the reader to [1, 28] regarding to Orlicz-Sobolev spaces. The usual norm on $L_{\Phi}(\Omega)$ is

$$\|u\|_{\Phi} = \inf \left\{ \lambda > 0 \mid \int_{\Omega} \Phi \left(\frac{u(x)}{\lambda} \right) dx \leq 1 \right\},$$

called as Luxemburg norm, and the norm on $W^{1,\Phi}(\Omega)$ is

$$\|u\|_{1,\Phi} = \|u\|_{\Phi} + \sum_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|_{\Phi},$$

known as Orlicz-Sobolev norm, while $W_0^{1,\Phi}(\Omega)$ stands for the closure of $C_0^\infty(\Omega)$ with respect to the norm in $W^{1,\Phi}(\Omega)$. We also recall that, under hypotheses $(\Phi_1) - (\Phi_3)$, the functions Φ and $\tilde{\Phi}$ are N-functions satisfying the Δ_2 -condition (see [28, pg 22]). As consequence of these, we have $L_{\Phi}(\Omega)$ and $W^{1,\Phi}(\Omega)$ are separable, reflexive, and Banach spaces.

Remark 5.1 *It is well known that (ϕ_3) implies that*

$$(\phi_3)' \quad \ell \leq \frac{\phi(t)t^2}{\Phi(t)} \leq m, \quad t > 0,$$

is verified. Furthermore, these hypotheses imply that $\Phi, \tilde{\Phi} \in \Delta_2$.

The inequality

$$\int_{\Omega} \Phi(u) dx \leq \int_{\Omega} \Phi(2\bar{d}|\nabla u|) dx \text{ for all } u \in W_0^{1,\Phi}(\Omega),$$

where $\bar{d} > 0$ is the diameter of Ω , is known as Poincaré's inequality (see e.g. [18]), and as a consequence of it, we have

$$\|u\|_{\Phi} \leq 2\bar{d}\|\nabla u\|_{\Phi} \text{ for all } u \in W_0^{1,\Phi}(\Omega),$$

that is, $\|u\| := \|\nabla u\|_{\Phi}$ defines an equivalent norm to the $\|\cdot\|_{1,\Phi}$ in $W_0^{1,\Phi}(\Omega)$.

Now, let Φ_* be the inverse of the function

$$t \in (0, \infty) \mapsto \int_0^t \frac{\Phi^{-1}(s)}{s^{\frac{N+1}{N}}} ds,$$

which is extended to \mathbb{R} by $\Phi_*(t) = \Phi_*(-t)$ for $t \leq 0$. We say that an N-function Ψ grows essentially more slowly than Φ_* , we denote this by $\Psi << \Phi_*$, if

$$\lim_{t \rightarrow \infty} \frac{\Psi(\lambda t)}{\Phi_*(t)} = 0 \text{ for all } \lambda > 0.$$

The imbeddings below (see [1]) was used in this paper

$$W_0^{1,\Phi}(\Omega) \xrightarrow{cpt} L_{\Psi}(\Omega) \text{ if } \Psi << \Phi_*,$$

and in particular, as $\Phi << \Phi_*$ (see [19, Lemma 4.14]), we have

$$W_0^{1,\Phi}(\Omega) \xrightarrow{cpt} L_{\Phi}(\Omega), \text{ and } W_0^{1,\Phi}(\Omega) \xrightarrow{\text{cont}} L_{\Phi_*}(\Omega).$$

It is worth to note that if just $(\phi_1) - (\phi_2)$, and $(\phi_3)'$ are verified, then

$$L_{\Phi}(\Omega) \xrightarrow{\text{cont}} L^{\ell}(\Omega)$$

is true, see [7, Lemma. D.2].

Below, we state some Lemmas whose proofs can be find in [11].

Lemma 5.1 Assume ϕ satisfies $(\phi_1) - (\phi_3)$. Set

$$\zeta_0(t) = \min\{t^\ell, t^m\}, \text{ and } \zeta_1(t) = \max\{t^\ell, t^m\}, \quad t \geq 0.$$

Then Φ satisfies

$$\begin{aligned} \zeta_0(t)\Phi(\rho) &\leq \Phi(\rho t) \leq \zeta_1(t)\Phi(\rho), \quad \rho, t > 0, \\ \zeta_0(\|u\|_\Phi) &\leq \int_\Omega \Phi(u)dx \leq \zeta_1(\|u\|_\Phi), \quad u \in L_\Phi(\Omega). \end{aligned}$$

Lemma 5.2 Assume that ϕ satisfies $(\phi_1) - (\phi_3)$. Set

$$\zeta_2(t) = \min\{t^{\tilde{\ell}}, t^{\tilde{m}}\}, \text{ and } \zeta_3(t) = \max\{t^{\tilde{\ell}}, t^{\tilde{m}}\}, \quad t \geq 0,$$

where $1 < \ell, m < N$, $\tilde{m} = m/(m-1)$, and $\tilde{\ell} = \ell(\ell-1)$. Then

$$\tilde{\ell} \leq \frac{t^2 \tilde{\Phi}'(t)}{\tilde{\Phi}(t)} \leq \tilde{m}, \quad t > 0, \quad \zeta_2(t)\tilde{\Phi}(\rho) \leq \tilde{\Phi}(\rho t) \leq \zeta_3(t)\tilde{\Phi}(\rho), \quad \rho, t > 0,$$

and

$$\zeta_2(\|u\|_{\tilde{\Phi}}) \leq \int_\Omega \tilde{\Phi}(u)dx \leq \zeta_3(\|u\|_{\tilde{\Phi}}), \quad u \in L_{\tilde{\Phi}}(\Omega).$$

Lemma 5.3 Let Φ be an N -function satisfying Δ_2 condition. Let $(u_n) \subset L_\Phi(\Omega)$ be a sequence such that $u_n \rightarrow u$ in $L_\Phi(\Omega)$. Then there is a subsequence $(u_{n_k}) \subseteq (u_n)$ such that:

- (i) $u_{n_k}(x) \rightarrow u(x)$ a.e. $x \in \Omega$,
- (ii) there is an $h \in L_\Phi(\Omega)$ such that $|u_{n_k}| \leq h$ a.e. in Ω .

Proof (Sketch): Since $L_\Phi(\Omega) \hookrightarrow L^1(\Omega)$, (see [1]), passing to a subsequence if necessary, we have $u_n \rightarrow u$ a.e. in Ω . Moreover, since

$$\int_\Omega \Phi(u_n - u)dx \rightarrow 0,$$

there exists an $\tilde{h} \in L^1(\Omega)$ such that $\Phi(u_n - u) \leq \tilde{h}$ a.e. in Ω .

Now, by using that Φ is convex, increasing and satisfies Δ_2 condition, we have

$$\Phi(|u_n|) \leq C\Phi\left(\frac{|u_n - u| + |u|}{2}\right) \leq \frac{C}{2} [\Phi(|u_n - u|) + \Phi(|u|)] \leq \frac{C}{2} [\tilde{h} + \Phi(|u|)].$$

Defining $h = \Phi^{-1}(C/2(\tilde{h} + \Phi(|u|)))$, it follows from $\tilde{h} \in L^1(\Omega)$ and $\Phi(|u|) \in L^1(\Omega)$, that

$$\begin{aligned} \int_\Omega \Phi(h)dx &= \int_\Omega \Phi\left(\Phi^{-1}\left(\frac{K}{2}(\tilde{h} + \Phi(|u|))\right)\right)dx \\ &= \int_\Omega \left(\frac{K}{2}(\tilde{h} + \Phi(|u|))\right)dx < \infty, \end{aligned}$$

showing that $h \in L_\Phi(\Omega)$. □

References

- [1] R.A. Adams and J.F. Fournier, *Sobolev Spaces*, Academic Press, New York, (2003).
- [2] C.O. Alves, F.J.S.A. Corrêa and J.V.A. Goncalves, *Existence of Solutions for Some Classes of Singular Hamiltonian Systems*, Adv. Nonlinear Stud., 5 (2005), 265-278.
- [3] H. Brézis and L. Oswald, *Remarks on sublinear elliptic equations*, Nonlinear Anal., 10 (1986), 55-64.
- [4] F. E. Browder, *Fixed point theory and nonlinear problems*, Bull. Amer. Math. Soc. (N.S.), 9 (1983), 1-39.
- [5] M.L. Carvalho, J.V. Goncalves and E.D. da Silva, *On quasilinear elliptic problems without the Ambrosetti Rabinowitz condition*, J. Math. Anal. Appl., 426 (2015), 466-483.
- [6] Y. S. Z. Choi and J. McKenna, *A singular Gierer-Meinhardt system of elliptic equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 17 (2000), 503-522.
- [7] P. Clément, B. de Pagter, G. Sweers and de F. Thlin, *Existence of solutions to a semilinear elliptic system through Orlicz-Sobolev spaces*, Mediterr. J. Math., 1(3) (2004), 241-267.
- [8] M.G. Crandall, P.H. Rabinowitz and L. Tartar, *On a Dirichlet problem with a singular nonlinearity*, Comm. Partial Differential Equations, 2 (1977), 193-222.
- [9] J.I. Díaz and J.E. Saa, *Existence et unicité de solutions positives pour certaines equations elliptiques quasilinéaires*, C. R. Acad. Sci., 305 Série I (1987), 521-524.
- [10] D.G. Figueiredo, *Lectures on the Ekeland variational principle with applications and detours*, Tata Inst. Fund. Res. Stud. Math., **81**, Springer-Verlag, (1989).
- [11] N. Fukagai, M. Ito and K. Narukawa, *Positive solutions of quasilinear elliptic equations with critical Orlicz-Sobolev nonlinearity on \mathbb{R}^N* , Funkcial. Ekvac., 49 (2006), 235-267.
- [12] N. Fukagai and K. Narukawa, *On the existence of multiple positive solutions of quasilinear elliptic eigenvalue problem*, Ann. Mat. Pura Appl., 186(4) (2007), 539-564.
- [13] M. Ghergu, *Lane-Emden systems with negative exponents*, J. Funct. Anal., 258 (2010), 3295-3318.
- [14] M. Ghergu and V. Radulescu, *Sublinear singular elliptic problems with two parameters*, J. Differential Equations, 195 (2003), 520-536.
- [15] J. Giacomoni, I. Schindler and P. Takac, *Singular quasilinear elliptic systems and Hölder regularity*, Adv. Differential Equations, 20(3-4) (2015), 259-298.
- [16] J. Giacomoni, I. Schindler and P. Takac, *Sobolev versus Hölder local minimizers and existence of multiple solutions for a singular quasilinear equation*, Ann. Sc. Norm. Super. Pisa Cl. Sci., 6 (2007), 117-158.

- [17] J.V. Goncalves, M. Rezende and C.A. Santos, *Positive solutions for a mixed and singular quasilinear problem*, Nonlinear Anal., 74 (2011), 132-140.
- [18] Gossez, J. P., *Orlicz-Sobolev spaces and nonlinear elliptic boundary value problems*, Nonlinear analysis, function spaces and applications, (Proc. Spring School, Horni Bradlo, 1978), Teubner, Leipzig, (1979) 59-94.
- [19] J.P. Gossez, *Nonlinear elliptic boundary value problems for equations with rapidly (or slowly) increasing coefficients*, Trans. Amer. Math. Soc., 190 (1974), 163-205.
- [20] D. D. Hai, *Singular elliptic systems with asymptotically linear nonlinearities*, Differential Integral Equations, 26(7-8) (2013), 837-844.
- [21] J. Hernández, F.J. Mancebo and J.M. Vega, *Positive solutions for singular semilinear elliptic systems*, Adv. Differential Equations, 13 (2008), 857-880.
- [22] J. Huentutripay and R. Manasevich, *Nonlinear Eigenvalues for a Quasilinear Elliptic System in Orlicz-Sobolev Spaces*, J. Dynam. Differential Equations, 18 (2006), 901-929.
- [23] A.C. Lazer and P.J. McKenna, *On a singular nonlinear elliptic boundary value problem*, Proc. AMS, 111 (1991), 721-730.
- [24] J.L. Lions, *Quelques methodes de resolution des problèmes aux limites non linéaires*, Dunod, Gauthier-Villars, (Paris) (1969).
- [25] A. Mohammed, *Positive solutions of the p -Laplace equation with singular nonlinearity*, J. Math. Anal. Appl., 352 (2009), 234-245.
- [26] J.R. Munkres, *Topology*, Person Education, New Jersey, (2000).
- [27] W.N. Ni, *Diffusion, cross-diffusion and spike-layer steady states*, Notices Amer. Math. Soc., 45 (1998), 9-18.
- [28] M. N. Rao and Z. D. Ren, *Theory of Orlicz Spaces*, Marcel Dekker, New York, (1985).
- [29] Z. Tan and F. Fang, *Orlicz-Sobolev versus Hölder local minimizer and multiplicity results for quasilinear elliptic equations*, J. Math. Anal. Appl., 402 (2013), 348-370.